High-dimensional McKean–Vlasov diffusion and the well-posedness of kinetic models of dilute polymers

Navier–Stokes–Fokker–Planck systems

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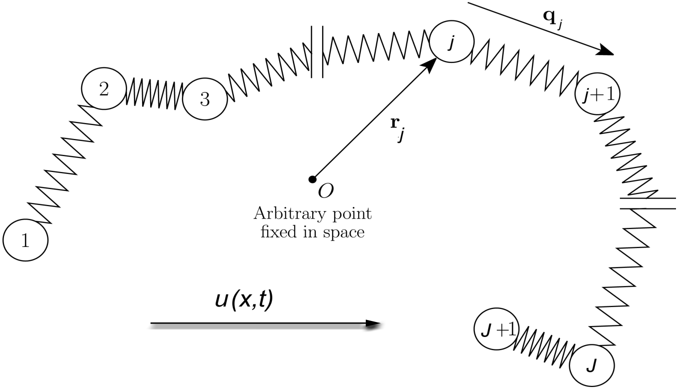
Bead-spring-chain model in a moving fluid

A polymer molecule is modelled as a linear chain of *massless* beads linked with elastic springs, subjected to Brownian noise, in a moving fluid: ⇝ NS–FP system.



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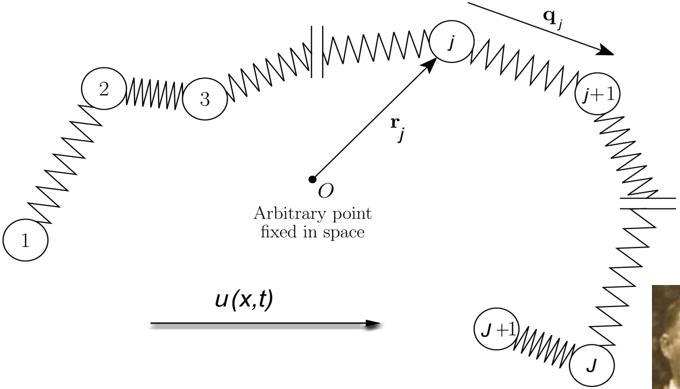

H. A. Kramers. Het gedrag van macromoleculen in een stroomende vloeistof, Physica 11 (1944), 1–19. [⟵ Werner Kuhn (1934), J.J. Hermans (1943)]

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Implication: Well-posedness of the Oldroyd-B model in $d = 3$ space dimensions?

Main results

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Conclusions:

- If the flow domain Ω is bounded, then the configuration space domain $D^J = \underbrace{D \times \cdots \times D}_{J}$, where $D = \Omega - \Omega = \{r - \hat{r} \,:\, r, \hat{r} \in \Omega\}$, is also bounded.

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- Before taking the small-mass limit, the Fokker–Planck equation, posed on $\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times [0, T] \ni (r, v, t)$, is mixed parabolic-hyperbolic.

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- After taking the small-mass limit, the Fokker–Planck equation, posed on $\Omega \times D^J \times [0, T] \ni (x, q, t)$, is parabolic.
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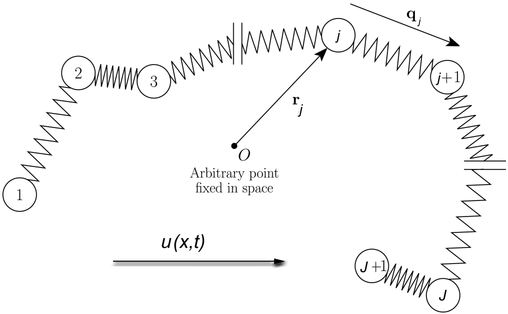
  $\Rightarrow$ FP eq. has center-of-mass diffusion $\Rightarrow$ Oldroyd-B has stress-diffusion
- We rigorously prove an assertion, deduced by Schieber & Öttinger (1988) using formal asymptotics, that:

  passage to the small-mass limit $\Rightarrow$ equilibration in momentum space.

Formulation of the model



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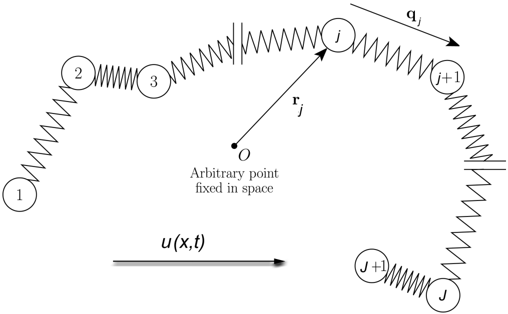

$$r := (r_1^{\mathrm{T}}, \dots, r_{J+1}^{\mathrm{T}})^{\mathrm{T}}, \quad r_j \in \Omega \quad \text{for } j = 1, \dots, J+1,$$
$$v := (v_1^{\mathrm{T}}, \dots, v_{J+1}^{\mathrm{T}})^{\mathrm{T}}, \quad v_j \in \mathbb{R}^d \quad \text{for } j = 1, \dots, J+1,$$
$$q = q(r) := (q_1^{\mathrm{T}}, \dots, q_J^{\mathrm{T}})^{\mathrm{T}}, \qquad q_j = q_j(r) := r_{j+1} - r_j \quad \text{for } j = 1, \dots, J.$$

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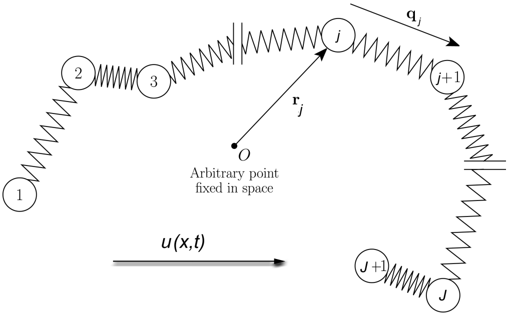

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$$q_j \in D := \Omega - \Omega = \{r - \hat{r} \,:\, r, \hat{r} \in \Omega\}, \qquad \text{for } j = 1, \ldots, J;$$

Condition: $$x := \frac{1}{J+1} \sum_{j=1}^{J+1} r_j.$$

Hydrodynamic interaction: Oseen system

On $\overline{\Omega} \times [0,T]$, where Ω is a bounded open convex $\mathcal{C}^2$ domain in $\mathbb{R}^d$, $d \in \{2,3\}$, $0 \in \Omega$, $b \in L^\infty(\Omega \times (0,T))$, $\nabla \cdot b = 0$:

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\begin{aligned}
\partial_t u + (b \cdot \nabla) u - \mu \triangle u + \nabla \pi &= \nabla \cdot \mathbb{K} && \text{for } (x,t) \in \Omega \times (0,T], \\
\nabla \cdot u &= 0 && \text{for } (x,t) \in \Omega \times (0,T], \\
u(x,t) &= 0 && \text{for } (x,t) \in \partial\Omega \times (0,T], \\
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with the polymeric extra-stress tensor (Kramers–Kirkwood stress tensor)

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\mathbb{K}(x,t;\varrho) := \sum_{j=1}^{J} \mathbb{E}^x \left(\lambda q_j \otimes q_j \right) \qquad \text{for } (x,t) \in \Omega \times (0,T],\ J \geq 1,
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$$
\mathcal{U}(r,t;\varrho) := \left(u(r_1,t;\varrho)^{\mathrm{T}}, \cdots, u(r_{J+1},t;\varrho)^{\mathrm{T}} \right)^{\mathrm{T}}.
$$

SDE:

$$\varepsilon^2 \ddot{r} = \underbrace{\mathcal{L}r}_{\text{Hookean spring force}} + \underbrace{\zeta\big(\mathcal{U}(r,t;\varrho) - \dot{r}\big)}_{\text{Stokes drag force}} + \underbrace{\sqrt{2\beta}\,\dot{W}.}_{\text{Brownian force}}$$

$\varepsilon^2 > 0$ is the mass of a bead in the chain, $\beta = k\mathtt{T}\zeta > 0$, where k is the Boltzmann constant, $\mathtt{T}$ is the absolute temperature and ζ is the drag coefficient; $\mathcal{L}$ is the following $(J+1) \times (J+1)$ block-matrix, called the Rouse matrix

$$\mathcal{L} := \lambda \begin{pmatrix} -\mathbb{I} & \mathbb{I} & \mathbb{O} & \dots & \mathbb{O} \\ \mathbb{I} & -2\mathbb{I} & \mathbb{I} & \ddots & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & -2\mathbb{I} & \mathbb{I} & \mathbb{O} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \mathbb{O} & \dots & \mathbb{I} & -2\mathbb{I} & \mathbb{I} \\ \mathbb{O} & \dots & \mathbb{O} & \mathbb{I} & -\mathbb{I} \end{pmatrix} \in \mathbb{R}^{(J+1)d\times(J+1)d},$$

where $\lambda > 0$ is a constant stiffness of the Hookean springs. W.l.o.g., we set $\zeta = 1$.

The SDE may then be rewritten as the first-order system

$$\begin{aligned} \varepsilon \dot{r} &= v, \\ \varepsilon \dot{v} &= \mathcal{L} r + \mathcal{U}(r, t; \varrho) - \varepsilon^{-1} v + \sqrt{2\beta}\, \dot{W}. \end{aligned}$$

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Let

$$\varrho \,:\, (r, v, t) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times [0, T] \mapsto \varrho(r, v, t) \in \mathbb{R}_{\geq 0}$$

be the probability density function of the diffusion process (r, v).

The law of (r, v) depends on ϱ itself through the function $\mathcal{U}$, and it is therefore a McKean–Vlasov diffusion process.

The polymeric extra stress tensor $\mathbb{K}$

We define

$$\mathbb{E}\Big(\sum_{j=1}^{J}\lambda q_j\otimes q_j\Big)(t) := \int_{\Omega^{J+1}\times\mathbb{R}^{(J+1)d}}\Big(\sum_{j=1}^{J}\lambda q_j(r)\otimes q_j(r)\Big)\,\varrho(r,v,t)\,\mathrm{d}r\,\mathrm{d}v$$

and perform a change of variables, replacing integration over $r\in\Omega^{J+1}$ by integration over $(q,x)\in D^J\times\Omega$ via the linear bijection

$$(q,x)\in D^J\times\Omega \quad\mapsto\quad r=B(q,x)\in\Omega^{J+1}, \qquad |\text{Jacobian}[B(q,x)]|=1.$$

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Then, for $(x,t) \in \Omega \times (0,T]$, the polymeric extra stress tensor is:

$$\mathbb{K}(x,t;\varrho) = \mathbb{E}^x\left(\sum_{j=1}^{J} \lambda q_j \otimes q_j\right)(x,t) = \frac{\int_{D^J\times\mathbb{R}^{(J+1)d}} \left(\sum_{j=1}^{J} \lambda q_j \otimes q_j\right) \varrho\big(B(q,x),v,t\big)\,\mathrm{d}q\,\mathrm{d}v}{\int_{D^J\times\mathbb{R}^{(J+1)d}} \varrho\left(B(q,x),v,t\right)\mathrm{d}q\,\mathrm{d}v}$$

Fokker–Planck (FP) equation for ϱ

$$\partial_t \varrho - \frac{\beta}{\varepsilon^2} \left(\sum_{j=1}^{J+1} \partial_{v_j} \cdot (v_j\, \varrho) + \beta\, \partial_{v_j}^2 \varrho \right) + \frac{1}{\varepsilon} \left(\sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho + ((\mathcal{L} r)_j + u(r_j, t)) \cdot \partial_{v_j} \varrho \right) = 0$$

for all $(r, v, t) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T]$,

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Let

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with $\partial\Omega$ at the j-th position in this $(J+1)$-fold Cartesian product, and

$$\nu^{(j)}(r) := (0^{\mathrm{T}}, \ldots, 0^{\mathrm{T}}, (\nu(r_j))^{\mathrm{T}}, 0^{\mathrm{T}}, \ldots, 0^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{(J+1)d}.$$

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$$\partial_t \varrho - \frac{\beta}{\varepsilon^2}\left(\sum_{j=1}^{J+1} \partial_{v_j} \cdot (v_j\, \varrho) + \beta\, \partial^2_{v_j} \varrho\right) + \frac{1}{\varepsilon}\left(\sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho + ((\mathcal{L}r)_j + u(r_j,t)) \cdot \partial_{v_j} \varrho\right) = 0$$

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Specular boundary condition:

$\varrho(r,v,t) = \varrho(r, v_*^{(j)}, t)$ for all $(r,v,t) \in \partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times (0,T]$, with $v \cdot \nu^{(j)}(r) < 0$, where

$$v_*^{(j)} := v - 2(v \cdot \nu^{(j)}(r))\, \nu^{(j)}(r), \qquad j = 1, \ldots, J+1.$$

Existence of solutions in the case when $\varepsilon > 0$

Define the Maxwellian for $v \in \mathbb{R}^{(J+1)d}$:

$$M(v) := (2\pi\beta)^{-\frac{J+1}{2}} \exp(-|v|^2/2\beta)$$

and rescale:

$$\widehat{\varrho} := \frac{\varrho}{M} \quad \text{and} \quad \widehat{\varrho}_0 := \frac{\varrho_0}{M}.$$

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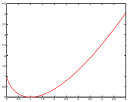
Consider the nonnegative strictly convex function with superlinear growth:

$$\mathcal{F}(s) := s(\log s - 1) + 1, \quad s \in \mathbb{R}_{>0}, \quad \text{with } \mathcal{F}(0) := 1.$$



Existence of solutions in the case when $\varepsilon > 0$

Define the Maxwellian for $v \in \mathbb{R}^{(J+1)d}$:

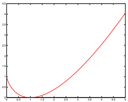
$$M(v) := (2\pi\beta)^{-\frac{J+1}{2}} \exp(-|v|^2/2\beta)$$

and rescale:

$$\widehat{\varrho} := \frac{\varrho}{M} \quad \text{and} \quad \widehat{\varrho}_0 := \frac{\varrho_0}{M}.$$

Consider the nonnegative strictly convex function with superlinear growth:

$$\mathcal{F}(s) := s(\log s - 1) + 1, \quad s \in \mathbb{R}_{>0}, \quad \text{with } \mathcal{F}(0) := 1.$$



The initial datum $\varrho_0 = \varrho_0(r, v) \geq 0$ is assumed to satisfy

$$\varrho_0 \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}), \qquad \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} \varrho_0(r, v)\,\mathrm{d}r\,\mathrm{d}v = 1,$$
$$M\mathcal{F}(\widehat{\varrho}_0) \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d});$$

i.e. $\varrho_0 \geq 0$ is assumed to have finite relative entropy with respect to M.

Existence of solutions to FP, for u fixed

STEP 1. For a given $u \in L^2(0,T;L^\infty(\Omega)^d)$ the Fokker–Planck equation has a nonnegative weak solution that satisfies the energy inequality:

$$\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\,\mathcal{F}(\widehat{\varrho}(t))\,\mathrm{d}v\,\mathrm{d}r + \frac{2\beta^2}{\varepsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\,|\partial_{v_j} \sqrt{\widehat{\varrho}}\,|^2\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau$$

$$\leq \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\,\mathcal{F}(\widehat{\varrho}_0)\,\mathrm{d}v\,\mathrm{d}r + \frac{16d}{\beta}\,(J+1)\,[\mathsf{diam}(\Omega)]^2\,T + \frac{J+1}{\beta}\,\|u\|^2_{L^2(0,T;L^\infty(\Omega))}$$

Coupling to the Oseen system

STEP 2.

Iterative process: define the sequence of functions $(u^{(k)}, \widehat{\varrho}^{(k)})$, for $k = 1, 2, \dots,$ and let $k \to \infty$.

Coupling to the Oseen system

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Iterative process: define the sequence of functions $(u^{(k)}, \widehat{\varrho}^{(k)})$, for $k = 1, 2, \dots,$ and let $k \to \infty$.

We set $u^{(1)} \equiv 0$. Given a divergence-free $u^{(k)} \in L^2(0, T; W_0^{1,\sigma}(\Omega)^d)$, for some $k \geq 1$ and $\sigma > d$, we define $\widehat{\varrho}^{(k)}$ as the weak solution of the FP eq.:

$$M\partial_t \widehat{\varrho}^{(k)} - \frac{\beta^2}{\varepsilon^2} \left(\sum_{j=1}^{J+1} \partial_{v_j} \cdot (M \partial_{v_j} \widehat{\varrho}^{(k)}) \right)$$

$$+ \frac{1}{\varepsilon} \left(\sum_{j=1}^{J+1} M v_j \cdot \partial_{r_j} \widehat{\varrho}^{(k)} + ((\mathcal{L}r)_j + u^{(k)}(r_j, t)) \cdot \partial_{v_j} (M \widehat{\varrho}^{(k)}) \right) = 0,$$

for all $(r, v, t) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T]$,

$$\widehat{\varrho}^{(k)}(r, v, 0) = \widehat{\varrho}_0(r, v) \qquad \text{for all } (r, v) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d},$$

subject to a (weakly imposed) specular boundary condition w.r.t. r.

STEP 3.

By STEP 1, $\widehat{\varrho}^{(k)}$ obeys the energy inequality:

$$\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\,\mathcal{F}(\widehat{\varrho}^{(k)}(t))\,\mathrm{d}v\,\mathrm{d}r + \frac{2\beta^2}{\varepsilon^2}\sum_{j=1}^{J+1}\int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\,\left|\partial_{v_j}\sqrt{\widehat{\varrho}^{(k)}}\,\right|^2\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau$$

$$\leq \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\,\mathcal{F}(\widehat{\varrho}_0)\,\mathrm{d}v\,\mathrm{d}r + \frac{16d}{\beta}\,(J+1)\,[\mathrm{diam}(\Omega)]^2\,T + \frac{J+1}{\beta}\,\|u^{(k)}\|^2_{L^2(0,T;L^\infty(\Omega))},$$

and the sequence $u^{(k)}$ will be shown (in STEP 6) to satisfy:

$$\|u^{(k)}\|_{L^2(0,T;L^\infty(\Omega))} \leq C,$$

where C is a positive constant, independent of k.

STEP 4.

Define $(u^{(k+1)}, \pi^{(k+1)})$, with

$$u^{(k+1)} \in L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;W_0^{1,2}(\Omega)^d),$$
$$\pi^{(k+1)} \in \mathcal{D}'(0,T;L^2(\Omega)/\mathbb{R}),$$

as the weak solution of the Oseen system:

$$\begin{aligned}
\partial_t u^{(k+1)} + (b\cdot\nabla)u^{(k+1)} - \mu\triangle u^{(k+1)} + \nabla\pi^{(k+1)} &= \nabla\cdot\mathbb{K}^{(k)} && \text{for } (x,t)\in\Omega\times(0,T],\\
\nabla\cdot u^{(k+1)} &= 0 && \text{for } (x,t)\in\Omega\times(0,T],\\
u^{(k+1)}(x,0) &= u_0(x) && \text{for } x\in\Omega,
\end{aligned}$$

$u_0 \in W_0^{1-2/z,z}(\Omega)^d$, with $z = d+\vartheta$, $\vartheta\in(0,1)$, is divergence-free, and

$$\mathbb{K}^{(k)}(x,t) := \frac{\int_{D^J\times\mathbb{R}^{(J+1)d}} \left(\sum_{j=1}^J \lambda q_j\otimes q_j\right) M\,\widehat{\varrho}^{(k)}\big(B(q,x),v,t\big)\,\mathrm{d}q\,\mathrm{d}v}{\int_{D^J\times\mathbb{R}^{(J+1)d}} M\,\widehat{\varrho}^{(k)}\,\big(B(q,x),v,t\big)\,\mathrm{d}q\,\mathrm{d}v}.$$

STEP 5.

Clearly,

$$\|\mathbb{K}^{(k)}\|_{L^\infty(0,T;L^\infty(\Omega))} \leq \lambda\, d \max_{q \in D^J} \|q\|^2 =: C,$$

where C is a positive constant, independent of k. Thus, there exists a $\mathbb{K} \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{symm}}))$ (to be identified), such that

$$\mathbb{K}^{(k)} \to \mathbb{K} \qquad \text{weak}^* \text{ in } L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{symm}})) \text{ as } k \to \infty.$$

STEP 5.

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We would like to show that

$$\mathbb{K}(x,t) := \frac{\int_{D^J \times \mathbb{R}^{(J+1)d}} (\sum_{j=1}^{J} \lambda q_j \otimes q_j)\, M\, \widehat{\varrho}\big(B(q,x),v,t\big)\, \mathrm{d}q\, \mathrm{d}v}{\int_{D^J \times \mathbb{R}^{(J+1)d}} M\, \widehat{\varrho}\,\big(B(q,x),v,t\big)\, \mathrm{d}q\, \mathrm{d}v}$$

but this is far from trivial. [We shall return to this in STEPS 8–10.]

STEP 6.

By maximal regularity theory for the Stokes system [Koch–Solonnikov (2001)]: there exists a constant $C = C_\sigma > 0$ s.t.

$$\|u^{(k+1)}\|_{W_\sigma^{1,\frac{1}{2}}(Q_T)} \leq C \left(\|\mathbb{K}^{(k)} - b \otimes u^{(k+1)}\|_{L^\sigma(Q_T)} + \|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)} \right),$$

where $\sigma = \min(\hat{\sigma}, z) > d$, $\hat{\sigma} := 2 + \frac{4}{d}$, with $z = d + \vartheta$ for some $\vartheta \in (0,1)$,

$$W_\sigma^{1,\frac{1}{2}}(Q_T) := L^\sigma(0,T; W_0^{1,\sigma}(\Omega)^d) \cap W^{1/2,\sigma}(0,T; L^\sigma(\Omega)^d).$$

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$$W_\sigma^{1,\frac{1}{2}}(Q_T) := L^\sigma(0,T; W_0^{1,\sigma}(\Omega)^d) \cap W^{1/2,\sigma}(0,T; L^\sigma(\Omega)^d).$$

As $W_\sigma^{1,\frac{1}{2}}(Q_T) \hookrightarrow L^2(0,T; W_0^{1,\sigma}(\Omega)^d) \hookrightarrow L^2(0,T; L^\infty(\Omega))$, – by STEP 5:

$$\|u^{(k+1)}\|_{L^2(0,T;L^\infty(\Omega))} \leq C(1 + \|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)}).$$

STEP 7.

We deduce that

$$u^{(k)} \to u \qquad \text{weakly in } L^2(0,T;W_0^{1,\sigma}(\Omega)^d) \text{ as } k \to \infty, \qquad \sigma > d,$$

$$u^{(k)} \to u \qquad \text{weakly in } W^{1,2}(0,T;W^{-1,\sigma'}(\Omega)^d) \text{ as } k \to \infty,$$

$$u^{(k)} \to u \qquad \text{strongly in } L^2(0,T;\mathcal{C}^{0,\gamma}(\overline{\Omega})^d) \text{ as } k \to \infty, \qquad 0 < \gamma < 1 - \tfrac{d}{\sigma}, \quad \sigma > d,$$

where the last result follows, via the Aubin–Lions lemma, thanks to the compact embedding of $W_0^{1,\sigma}(\Omega)^d$ into $\mathcal{C}^{0,\gamma}(\overline{\Omega})^d$ for $0 < \gamma < 1 - \frac{d}{\sigma}$, $\sigma > d$.

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It is now straightforward to pass to the limit in the Oseen system:

$$\begin{aligned} \partial_t u + (b\cdot\nabla)u - \mu\triangle u + \nabla\pi &= \nabla\cdot\mathbb{K} && \text{for } (x,t)\in\Omega\times(0,T],\\ \nabla\cdot u &= 0 && \text{for } (x,t)\in\Omega\times(0,T],\\ u(x,t) &= 0 && \text{for } (x,t)\in\partial\Omega\times(0,T],\\ u(x,0) &= u_0(x) && \text{for } x\in\Omega. \end{aligned}$$

STEP 8.

Must identify the weak* limit $\mathbb{K}$ of the sequence $(\mathbb{K}^{(k)})_{k\geq 0}$ in terms of the limit $\widehat{\varrho}$ of the sequence $(\widehat{\varrho}^{(k)})_{k\geq 0}$.

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We rewrite the FP equation as

$$M\partial_t\widehat{\varrho}^{(k)} - \frac{\beta^2}{\varepsilon^2}\left(\sum_{j=1}^{J+1}\partial_{v_j}\cdot(M\partial_{v_j}\widehat{\varrho}^{(k)})\right) + \frac{1}{\varepsilon}\left(\sum_{j=1}^{J+1}Mv_j\cdot\partial_{r_j}\widehat{\varrho}^{(k)}\right)$$

$$= -\frac{1}{\varepsilon}\sum_{j=1}^{J+1}\Big(((\mathcal{L}r)_j + u^{(k)}(r_j,t))\cdot\partial_{v_j}(M\widehat{\varrho}^{(k)})\Big)$$

$$\text{in } \mathcal{D}'(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}\times(0,T)),$$

$$\widehat{\varrho}^{(k)}(r,v,0) = \widehat{\varrho}_0(r,v),$$

and note that the differential operator on the left-hand side is hypoelliptic, and the right-hand side is both bounded and equi-bounded in $L^1(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}\times(0,T))$.

STEP 9.

By an argument of R. DiPerna & P.-L. Lions (1988), it follows that

$$\widehat{\varrho}^{(k)} \to \widehat{\varrho} \qquad \text{strongly in } L^1(0,T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{as } k \to \infty.$$

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This then (eventually, after a further technical argument,) implies that

$$\mathbb{K}^{(k)}(x,t) := \frac{\int_{D^J \times \mathbb{R}^{(J+1)d}} \left(\sum_{j=1}^{J} \lambda q_j \otimes q_j\right) M\, \widehat{\varrho}^{(k)}\big(B(q,x), v, t\big)\, \mathrm{d}q\, \mathrm{d}v}{\int_{D^J \times \mathbb{R}^{(J+1)d}} M\, \widehat{\varrho}^{(k)}\, \big(B(q,x), v, t\big)\, \mathrm{d}q\, \mathrm{d}v}$$

converges to

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weakly* in $L^\infty(\Omega \times (0,T))$.

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weakly* in $L^\infty(\Omega\times(0,T))$. Thus we have shown the existence of a weak solution to the coupled Oseen–Fokker–Planck system with the bead-mass $\varepsilon>0$ fixed. □

Small-mass limit and equilibration in momentum space

We showed, for each $\varepsilon > 0$, the existence of $u = u_\varepsilon$ and $\widehat{\varrho} = \widehat{\varrho}_\varepsilon$, such that

$$u_\varepsilon \in \mathcal{C}([0,T]; L^\sigma(\Omega)^d) \cap L^2(0,T; W_0^{1,\sigma}(\Omega)^d) \cap W^{1,2}(0,T; W^{-1,\sigma}(\Omega)^d),$$

with $\sigma = \min(\hat{\sigma}, z) > d$, $\hat{\sigma} := 2 + \frac{4}{d}$ and $z = d + \vartheta$ for some $\vartheta \in (0,1)$, is a weak solution to the Oseen system,

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with $\sigma = \min(\hat{\sigma}, z) > d$, $\hat{\sigma} := 2 + \frac{4}{d}$ and $z = d + \vartheta$ for some $\vartheta \in (0,1)$, is a weak solution to the Oseen system, and $\widehat{\varrho}_\varepsilon$ with

$$\mathcal{F}(\widehat{\varrho}_\varepsilon) \in L^\infty(0,T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})),$$

$$\nabla_v \sqrt{\widehat{\varrho}_\varepsilon} \in L^2(0,T; L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})),$$
$$\nabla_v \widehat{\varrho}_\varepsilon \in L^2(0,T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})),$$
$$M\, \partial_t \widehat{\varrho}_\varepsilon \in L^2(0,T; (W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'), \quad s > (J+1)d + 1,$$

satisfies the following weak form of the Fokker–Planck equation:

$$
\begin{aligned}
&\int_0^t \left\langle M\,\partial_\tau \widehat{\varrho}_\varepsilon(\cdot,\cdot,\tau), \varphi(\cdot,\cdot,\tau)\right\rangle \mathrm{d}\tau \\
&\qquad + \frac{\beta^2}{\varepsilon^2}\left(\sum_{j=1}^{J+1}\int_0^t \int_{\Omega^{J+1}}\int_{\mathbb{R}^{(J+1)d}} M(v)\,\partial_{v_j}\widehat{\varrho}_\varepsilon \cdot \partial_{v_j}\varphi\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau\right) \\
&\qquad - \frac{1}{\varepsilon}\left(\sum_{j=1}^{J+1}\int_0^t \int_{\Omega^{J+1}}\int_{\mathbb{R}^{(J+1)d}} M(v)\,v_j\widehat{\varrho}_\varepsilon \cdot \partial_{r_j}\varphi\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau\right) \\
&\qquad - \frac{1}{\varepsilon}\left(\sum_{j=1}^{J+1}\int_0^t \int_{\Omega^{J+1}}\int_{\mathbb{R}^{(J+1)d}} M(v)\,((\mathcal{L}r)_j + u_\varepsilon(r_j,\tau))\,\widehat{\varrho}_\varepsilon \cdot \partial_{v_j}\varphi\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau\right) = 0 \\
&\forall\,\varphi \in L^2(0,T; W^{1,2}_{*,M}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}) \cap W^{s,2}_{*}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})), \quad s > (J+1)d+1.
\end{aligned}
$$

$$
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&\quad + \frac{\beta^2}{\varepsilon^2}\left(\sum_{j=1}^{J+1}\int_0^t \int_{\Omega^{J+1}}\int_{\mathbb{R}^{(J+1)d}} M(v)\,\partial_{v_j}\widehat{\varrho}_\varepsilon \cdot \partial_{v_j}\varphi \,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau\right) \\
&\quad - \frac{1}{\varepsilon}\left(\sum_{j=1}^{J+1}\int_0^t \int_{\Omega^{J+1}}\int_{\mathbb{R}^{(J+1)d}} M(v)\,v_j\widehat{\varrho}_\varepsilon \cdot \partial_{r_j}\varphi \,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau\right) \\
&\quad - \frac{1}{\varepsilon}\left(\sum_{j=1}^{J+1}\int_0^t \int_{\Omega^{J+1}}\int_{\mathbb{R}^{(J+1)d}} M(v)\,((\mathcal{L}r)_j + u_\varepsilon(r_j,\tau))\,\widehat{\varrho}_\varepsilon \cdot \partial_{v_j}\varphi \,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau\right) = 0 \\
&\forall\,\varphi \in L^2(0,T; W^{1,2}_{*,M}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}) \cap W^{s,2}_{*}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})), \quad s > (J+1)d+1.
\end{aligned}
$$

Furthermore $\widehat{\varrho}_\varepsilon(\cdot,\cdot,0) = \widehat{\varrho}_0(\cdot,\cdot)$,

$$
\int_{\Omega^{J+1}\times\mathbb{R}^{(J+1)d}} M\,\widehat{\varrho}_\varepsilon(r,v,t)\,\mathrm{d}r\,\mathrm{d}v = \int_{\Omega^{J+1}\times\mathbb{R}^{(J+1)d}} M\,\widehat{\varrho}_0(r,v)\,\mathrm{d}r\,\mathrm{d}v = 1 \qquad \forall\, t \in (0,T].
$$

In addition, $\widehat{\varrho}_\varepsilon$ satisfies the following energy inequality:

$$\int_{\Omega^{J+1}}\int_{\mathbb{R}^{(J+1)d}} M(v)\,\mathcal{F}(\widehat{\varrho}_\varepsilon(t))\,\mathrm{d}v\,\mathrm{d}r + \frac{2\beta^2}{\varepsilon^2}\sum_{j=1}^{J+1}\int_0^t\int_{\Omega^{J+1}}\int_{\mathbb{R}^{(J+1)d}} M(v)\,|\partial_{v_j}\sqrt{\widehat{\varrho}_\varepsilon}\,|^2\;\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau$$
$$\leq C\Big[1+\int_{\Omega^{J+1}}\int_{\mathbb{R}^{(J+1)d}} M(v)\,\mathcal{F}(\widehat{\varrho}_0)\,\mathrm{d}v\,\mathrm{d}r\Big],$$

where $C = C(\|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)}, \|b\|_{L^\infty(0,T;L^\infty(\Omega))})$, $\sigma = \min(\hat{\sigma}, z) > d$, $\hat{\sigma} := 2 + \frac{4}{d}$ and $z = d + \vartheta$ for some $\vartheta \in (0, 1)$; C is independent of $\varepsilon > 0$.

In addition, $\widehat{\varrho}_\varepsilon$ satisfies the following energy inequality:

$$\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\,\mathcal{F}(\widehat{\varrho}_\varepsilon(t))\,\mathrm{d}v\,\mathrm{d}r + \frac{2\beta^2}{\varepsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\,|\partial_{v_j}\sqrt{\widehat{\varrho}_\varepsilon}\,|^2\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau \\ \leq C\left[1 + \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\,\mathcal{F}(\widehat{\varrho}_0)\,\mathrm{d}v\,\mathrm{d}r\right],$$

where $C = C(\|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)}, \|b\|_{L^\infty(0,T;L^\infty(\Omega))})$, $\sigma = \min(\hat{\sigma}, z) > d$, $\hat{\sigma} := 2 + \frac{4}{d}$ and $z = d + \vartheta$ for some $\vartheta \in (0,1)$; C is independent of $\varepsilon > 0$.

Hence,

$$\begin{aligned} (\mathcal{F}(\widehat{\varrho}_\varepsilon))_{\varepsilon>0} &\quad \text{is bounded in } L^\infty(0,T;L^1_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})), \\ (\nabla_v\sqrt{\widehat{\varrho}_\varepsilon})_{\varepsilon>0} &\quad \text{is bounded in } L^2(0,T;L^2_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})), \\ (M\,\partial_t\widehat{\varrho}_\varepsilon)_{\varepsilon>0} &\quad \text{is bounded in } L^2(0,T;(W^{s,2}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}))'), \end{aligned}$$

for $s > (J+1)d + 1$.

STEP 10.

$$
\begin{aligned}
\widehat{\varrho}_\varepsilon &\rightharpoonup \widehat{\varrho}_{(0)} && \text{weakly in } L^p(0,T;L^1_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})) \qquad \forall\, p\in[1,\infty),\\
M\,\partial_t\widehat{\varrho}_\varepsilon &\rightharpoonup M\,\partial_t\widehat{\varrho}_{(0)} && \text{weakly in } L^2(0,T;(W^{s,2}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}))'), \quad s>(J+1)d+1,\\
v_j\,\widehat{\varrho}_\varepsilon &\rightharpoonup v_j\,\widehat{\varrho}_{(0)} && \text{weakly in } L^2(0,T;L^1_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})), \quad j=1,\dots,J+1.
\end{aligned}
$$

STEP 10.

$$\begin{aligned} \widehat{\varrho}_\varepsilon \rightharpoonup \widehat{\varrho}_{(0)} \quad & \text{weakly in } L^p(0,T;L^1_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})) \qquad \forall\, p\in[1,\infty),\\ M\,\partial_t\widehat{\varrho}_\varepsilon \rightharpoonup M\,\partial_t\widehat{\varrho}_{(0)} \quad & \text{weakly in } L^2(0,T;(W^{s,2}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}))'), \quad s>(J+1)d+1,\\ v_j\,\widehat{\varrho}_\varepsilon \rightharpoonup v_j\,\widehat{\varrho}_{(0)} \quad & \text{weakly in } L^2(0,T;L^1_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})), \quad j=1,\dots,J+1. \end{aligned}$$

Also, because

$$\|u_\varepsilon\|_{L^2(0,T;W^{1,\sigma}(\Omega))\cap W^{1,2}(0,T;W^{-1,\sigma}(\Omega))} \leq C(1+\|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)}),$$

with $\sigma > d$, whereby

$$u_\varepsilon \rightharpoonup u_{(0)} \quad \text{weakly in } L^2(0,T;W^{1,\sigma}(\Omega))\cap W^{1,2}(0,T;W^{-1,\sigma}(\Omega)),$$

$$u_\varepsilon \to u_{(0)} \quad \text{strongly in } L^2(0,T;\mathcal{C}^{0,\gamma}(\overline{\Omega})^d) \text{ as } \varepsilon\to 0_+, \qquad 0<\gamma<1-\tfrac{d}{\sigma}.$$

STEP 11.

Furthermore,

$$\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \left| \partial_{v_j} \sqrt{\widehat{\varrho}_{(0)}} \right|^2 \, \mathrm{d}v \, \mathrm{d}r \, \mathrm{d}\tau \leq 0.$$

STEP 11.

Furthermore,

$$\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \, |\partial_{v_j} \sqrt{\widehat{\varrho}_{(0)}}\,|^2 \, \mathrm{d}v \, \mathrm{d}r \, \mathrm{d}\tau \leq 0.$$

Hence,

$$\widehat{\varrho}_{(0)}(r,v,t) = \eta(r,t) \qquad \text{a.e. in } \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0,T)$$

with $\eta \in L^\infty(0,T; L^1(\Omega^{J+1}))$ to be determined.

STEP 11.

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Thus, we have proved equilibration in momentum space:

$$\varrho_{(0)}(r,v,t) := M(v)\, \widehat{\varrho}_{(0)}(r,v,t) = M(v)\, \eta(r,t),$$

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STEP 11.

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J. Schieber & H. C. Öttinger, The effects of bead inertia on the Rouse model. J. Chem. Phys. 89, no. 11, (1988).

STEP 12.

The small-mass limit of the coupled Oseen–Fokker–Planck system satisfies

$$
\begin{aligned}
\partial_t u_{(0)} + (b\cdot\nabla)u_{(0)} - \mu\triangle u_{(0)} + \nabla\pi_{(0)} &= \nabla\cdot\mathbb{K}_{(0)} && \text{for } (x,t)\in\Omega\times(0,T],\\
\nabla\cdot u_{(0)} &= 0 && \text{for } (x,t)\in\Omega\times(0,T],\\
u_{(0)}(x,t) &= 0 && \text{for } (x,t)\in\partial\Omega\times(0,T],\\
u_{(0)}(x,0) &= u_0(x) && \text{for } x\in\Omega,
\end{aligned}
$$

with $\mathbb{K}_{(0)}$ to be identified.

The identification of $\mathbb{K}_{(0)}$ (via the DIV-CURL Lemma) is (again) technical:

$$\mathbb{K}_{(0)}(x,t) := \frac{\int_{D^J} \sum_{j=1}^{J} (F(q_j) \otimes q_j)\, \eta\big(B(q,x),t\big)\, \mathrm{d}q}{\int_{D^J} \eta\big(B(q,x),t\big)\, \mathrm{d}q} \qquad \text{for } (x,t) \in \Omega \times (0,T],$$

where $B(x,q) = r$,

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where $B(x,q) = r$, and $\eta \geq 0$, with $\int_{\Omega^{J+1}} \eta(r,t)\, \mathrm{d}r = 1$ for all $t \in [0,T]$, solves

$$\partial_t \eta + \sum_{j=1}^{J+1} \partial_{r_j} \cdot \Big(\eta\left((\mathcal{L}r)_j + u_{(0)}(r_j,\cdot)\right)\Big) - \beta \sum_{j=1}^{J+1} \partial^2_{r_j} \eta = 0 \qquad \text{in } \Omega^{J+1} \times (0,T],$$

$$\eta(\cdot,0) = \widehat{\varrho}_0 \in L^1 \log L^1(\Omega^{J+1}; \mathbb{R}_{\geq 0}) \quad + \begin{cases} \text{zero-flux boundary condition on} \\ \partial\Omega^{(j)} \times (0,T] \text{ for } j = 1,\ldots,J+1. \end{cases}$$

STEP 13.

Change variables in FP from $r \in \Omega^{J+1}$ to $(x, q) \in \Omega \times D^J$.
Hence, $\psi(x, q, t) := \eta(B(q, x), t) = \eta(r, t)$ solves on $\Omega \times D^J \times [0, T]$:

$$\partial_t \psi + \nabla_x \cdot (u_{(0)} \psi) + \sum_{j=1}^{J} \partial_{q_j} \cdot ((\nabla_x u_{(0)}) q_j \psi)$$
$$- \beta \sum_{i,j=1}^{J} \partial_{q_j} \cdot \left[\mathcal{R}_{ij} \, \mathfrak{M}(q) \, \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) \right] - \frac{\beta}{J+1} \Delta_x \psi = 0,$$

STEP 13.

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$$- \beta \sum_{i,j=1}^{J} \partial_{q_j} \cdot \left[\mathcal{R}_{ij} \, \mathfrak{M}(q) \, \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) \right] - \frac{\beta}{J+1} \Delta_x \psi = 0,$$

with the initial condition $\psi(x, q, 0) = \psi_0(x, q) := \hat{\varrho}_0(B(q, x))$ and zero flux boundary conditions, where

$$\mathfrak{M}(q) := (2\pi\beta)^{-\frac{1}{2} Jd} \exp\left(-|q|^2/2\beta\right), \quad q \in D^J.$$

A. W. El-Kareh and L. G. Leal, Existence of solutions for all Deborah numbers for a non-Newtonian model modified to include diffusion. J. Non-Newtonian Fluid Mech., 33 (1989), pp. 257287.

Note:

If the initial datum ψ_0 is such that, for some constant $n > 0$,

$$\int_{D^J} \psi_0(x,q)\,\mathrm{d}q = n^{-1} \qquad \text{for a.e. } x \in \Omega,$$

then it follows that

$$\int_{D^J} \eta(B(q,x),t)\,\mathrm{d}q = \int_{D^J} \psi(x,q,t)\,\mathrm{d}q = n^{-1} \qquad \text{for a.e. } (x,t) \in \Omega \times [0,T],$$

so the original expression for $\mathbb{K}_{(0)}$ simplifies to the Kramers' expression:

$$\mathbb{K}_{(0)}(x,t) = n \int_{D^J} \sum_{j=1}^{J} (F(q_j) \otimes q_j)\,\psi(x,q,t)\,\mathrm{d}q.$$

The number $n > 0$ is called the *polymer number density per unit volume.*

J W Barrett and E Süli. Existence and equilibration of global weak solutions to kinetic models for dilute polymers I: finitely extensible nonlinear bead-spring chains. M3AS 21 (6), 1211-1289, 2011. DOI: 10.1142/S0218202511005313.

M Dostalík, J Málek, V Pruša, and E Süli. A simple approach to thermodynamically consistent modelling of non-isothermal flows of dilute compressible polymeric fluids. Fluids 2020, Volume 5, Issue 3, 29 pp.; DOI:10.3390/fluids5030133.

E. Süli and G. Yahiaoui.
McKean-Vlasov diffusion and the well-posedness of the Hookean bead-spring-chain model for dilute polymeric fluid: small-mass limit and equilibration in momentum space. arXiv.